APPENDIX A

This appendix gives the proofs of the equations given in Chapter 3.

- **Proof of Eqn (3.3)**

$$w_k = \sum_j w_j q_{jk}$$

From Eqn (3.1) ,we have

$$w_k = \Pr\{W_i = k\}$$

$$= \sum_j \Pr\{W_i = k, W_{i-1} = j\}$$ as dependence between delays is Markovian

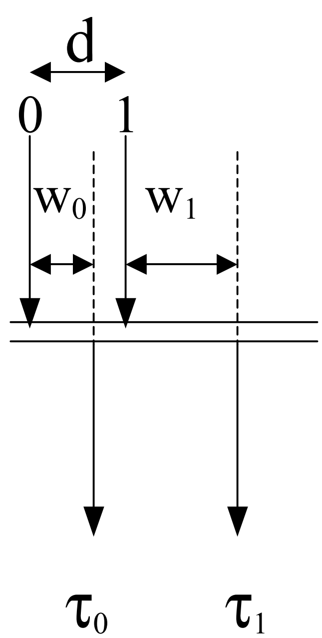
$$= \sum_j \Pr\{W_i = k / W_{i-1} = j\}\Pr\{W_{i-1} = j\}$$ writing the above expn. in terms of conditional probabilities

$$= \sum_j q_{jk} w_j$$ using (3.1) and (3.2)

- **Proof of Eqn(3.6)**

$$f_n(k) = \sum_{i \geq 0} w_i q^{(n)}_{i,i+k}$$

From the following figure we see that : $U_1+d=W_1+d-W_0 \Rightarrow U_1=W_1-W_0$



Similarly from Fig.3.1, we see that $U_n=W_n-W_0$

Eqn (3.5) gives, $f_n(k)=\Pr\{U_n=k\}$

Using the expression for U_n derived above : $f_n(k)=\Pr\{W_n-W_0=k\}$

$=\Pr\{W_n=k+W_0\}$

writing this in terms of the joint probability we get

$$f_n(k)=\sum_{i\geq 0}\Pr\{W_n=k+W_0, W_0=i\}$$

i.e,

$$f_n(k)=\sum_j \Pr\{W_n=k+W_0/\ W_0=i\}\Pr\{W_0=i\}$$

i.e,

$$f_n(k)=\sum_j \Pr\{W_n=k+i\ /\ W_0=i\}\Pr\{W_0=i\}$$

using (3.1) and (3.2)

$$f_n(k)=\sum_j w_i q^{(n)}_{i,i+k}$$

where, $q^{(n)}_{i,j}$ is the i-j component of the nth power of the transition matrix, q_{jk}, .

- **Proof of Eqn (3.10)**

$$Q(j,k)=\sum_n P_n(j,k)\frac{(\lambda d)^n}{n!}e^{-\lambda d}$$

Eqn (3.7) gives the definition of Q(j,k) i.e, $Q(j,k) = \Pr\{W_i>k/W_{i-1}=j\}$

writing this in terms of the joint probability we get

$$Q(j,k)=\sum_n \Pr\{W_i>k, \text{n Poisson arrivals in } ((i-1)d, id)/\ W_{i-1}=j\}$$

The following sets of equations are obtained using the relation between joint probability and conditional probability.

$$Q(j,k)=\frac{\sum_n \Pr\{W_i>k, \text{n Poisson arrivals in } ((i-1)d, id),\ W_{i-1}=j\}}{\Pr\{W_{i-1}=j\}}$$

$$=\frac{\sum_n \Pr\{W_i>k/W_{i-1}=j, \text{n Poisson arrivals in } ((i-1)d, id)\}\Pr\{\text{n Poisson arrivals in } ((i-1)d, id), W_{i-1}=j\}}{\Pr\{W_{i-1}=j\}}$$

$$=\sum_n \Pr\{W_i>k/W_{i-1}=j, \text{n Poisson arrivals in } ((i-1)d, id)\}\Pr\{\text{n Poisson arrivals in } ((i-1)d, id)/W_{i-1}=j\}$$

Using Eqn (3.8) and the fact that the Poisson arrivals is independent of the delay of the $i\text{-}1^{th}$ cell we have

$$Q(j,k) = \sum_n P_n(j,k)\frac{(\lambda d)^n}{n!}e^{-\lambda d}$$

- **Proof of Eqn (3.11)**

$$P_n(j,k) = \begin{cases} 0 & \text{for } (j+1 \geq d \text{ and } n \leq d+k-j-1) \text{ or } (j+1 < d \text{ and } n \leq k) \\ \sum_{s=1}^{n-k} \binom{n}{s+k}\left(\frac{s}{d}\right)^{s+k}\left(1-\frac{s}{d}\right)^{n-s-k}\frac{d-n+k}{d-s} & \text{for } j+1 < d \text{ and } k < n \leq d+k-j-1 \\ 1 & \text{for } n > d+k-j-1 \end{cases}$$

As explained in Sec3.1, since the service discipline is assumed to be FIFO, W_i is identical to the queue length L_i , as seen by the i^{th} periodic cell.
Consider the following scenario:
In between arrival of two CBR cells ((i-1)d ,id) there are d slots.
Let delay of the $i\text{-}1^{th}$ cell be j ie, $L_{i-1}=j$
Let $j \geq d-1$
Then, in the d-1 slots before the i^{th} CBR packet arrives no. of cells serviced from among these j packets is j-(d-1).

Hence if the no. of Poisson cells arriving in this interval(n_i) is $\leq k+d-j-1$, then since $L_i = n_i + l_{i-1} - (d-1)$, we have,

$L_i \leq (k+d-j-1) + (j-d+1) \Rightarrow L_i \leq k \Rightarrow P_n(j,k) = 0$

Consider the other case $j \leq d-1$, then in the d-1 slots before the next CBR cell arrives all these are serviced .
Therefore if $n_i \leq k \Rightarrow L_i \leq k \Rightarrow P_n(j,k) = 0$

Hence $P_n(j,k) = 0$ for $j \geq d-1$ and $n \leq k+d-j-1$
or $j \leq d-1$ and $n \leq k$

Reversing the inequalities in the j and n ranges in the above expressions we get

$P_n(j,k) = 1$ for $j+1 < d$ and $n \geq k+d-j-1$
or $j+1 \geq d-1$ and $n > k$

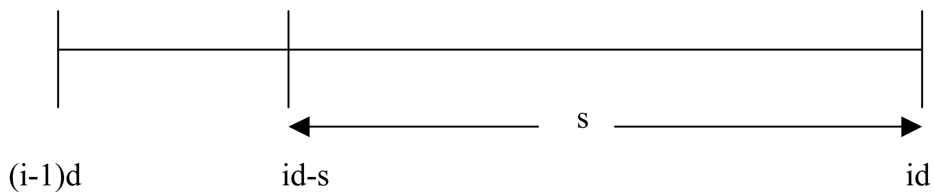
The above covers the entire range of j and hence the range can simply be written in terms of n alone as

$P_n(j,k) = 1$ for $n \geq k+d-j-1$

This proves the first part and last part of Eqn.(3.11) .

Now we shall prove the intermediate case ie in the range $j+1<d$ and $k<n<d+k-j-1$

Let $\nu_n(k)$ be the complementary distribution of the queue length at service instant id due to only the n Poisson arrivals(ie, discounting the cells already present at (i-1)d).$P_n(j,k)$ is exactly equal to $\nu_n(k)$ iff the queue is empty at any service instant in the interval ((i-1)d,id). Consider the following schematic:



$$\nu_n(k) = \Pr\{\text{queue empty at id-s}\}$$
$$= \sum_{s=1}^{n-k} \Pr\{\text{queue empty at id - s, k + s arrivals in (id - s, id)}\}$$
$$= \sum_{s=1}^{n-k} \Pr\{\text{queue empty at id - s/ k + s arrivals in (id - s, id)}\}\Pr\{\text{k + s arrivals in (id - s, id)}\}$$

Now, we first find Pr{k+s arrivals in (id-s,id)}.
Poisson arrivals are uniformly distributed over any finite interval.
Hence Probability of having exactly (k+s) arrivals in s slots is

$$\left(\frac{s}{d}\right)^{s+k}\left(1-\frac{s}{d}\right)^{n-s-k}$$

Since we have to have a total of n arrivals , if k+s arrivals take place in s slots(1st term in the above expn.) then remaining n-k-s arrivals have to take place in the remaining d-s slots(last term in the above expn.). Since we can take any k+s arrivals from the n arrivals we add a combinatorial term to the above expn. Hence

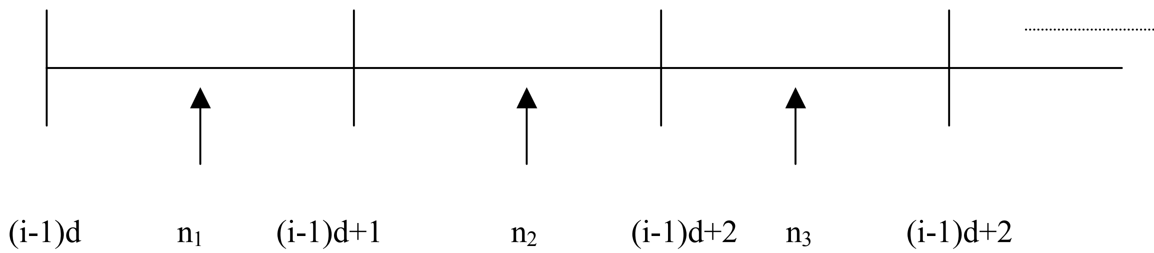
$$\Pr\{\text{k+s arrivals in (id-s,id)}\}.= \binom{n}{s+k}\left(\frac{s}{d}\right)^{s+k}\left(1-\frac{s}{d}\right)^{n-s-k}$$

Now we find Pr{queue empty at id-s/k+s arrivals in (id-s,id)}

$$\Pr\{\text{queue empty at } id-s/k+s \text{ arrivals in } (id-s,id)\} = \Pr\{\text{queue empty at } id-s/n-k-s \text{ arrivals in } ((i-1)d,id-s)\}$$

Define n_l as the no. of Poisson arrivals in $((i-1)d+l-1,(i-1)d+l)$

Diagramatically the above can be viewed as



Let $N_l = n_1 + n_2 + n_3 + \ldots n_l$

If the queue should be empty at id-s, No of arrivals in the first d-s slots should be less than d-s(as it is a synchronous server, at every time slot a service takes place and a cell is evicted). There we can write Pr{queue empty at id-s/n-k-s arrivals in ((i-1)d,id-s)} as

$$\Pr\{\text{queue empty at } id-s/n-k-s \text{ arrivals in } ((i-1)d,id-s)\}=\Pr\{N_l<l, l=1,2,\ldots d-s/N_{d-s}<n-k-s\}$$

Using Theorem 1,Page 10 in [LT67] we can write

$$\begin{aligned}\Pr\{N_l<l, l=1,2,\ldots d-s/N_{d-s}<n-k-s\} &= (d-s-[n-k-s])/(d-s)\\ &= (d-n+k)/(d-s)\end{aligned}$$